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Two Applications of Percolation to Cellular Automata

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The point of this paper is to show how ideas from percolation can be used to study the asymptotic behavior of some cellular automata systems. In particular, using these ideas, we prove that the Greenberg–Hastings and cyclic cellular automata models with three colors, threshold 2, and the L^∞ neighborhood are uniformly asymptotically locally periodic in $d \geqslant 2$ dimensions. We also show that every lattice point is eventually "controlled by a finite clock" in the standard Greenberg–Hastings and cyclic cellular automata models in two dimensions, which is a stronger description than the already known asymptotic behavior.

KEY WORDS: Cellular automata; percolation.

1. INTRODUCTION

We begin by describing a large class of models which can be collectively called generalized Greenberg–Hastings (GH) and cyclic cellular automata (CCA) models. All of these models will have as their state space $X = \{0, 1,..., k-1\}^{\mathbf{Z}^d}$, where k is considered to be the number of colors in the model. In order to define these models, one needs to specify three parameters (in addition to the dimension d), which are, respectively, the number of colors k, the neighborhood set $\mathcal{N}$ containing 0, and the threshold level θ, a positive integer. In all cases, these models will be a continuous (in the product topology) translation-invariant mapping from X of itself, something which is usually called a *cellular automaton*. Throughout this paper, we will always let η_n denote the configuration at time n, with of course η_0 being the initial configuration.

The generalized GH models are extensions of the standard GH model, which was studied nonrigorously in refs. 14 and 16 and rigorously in refs.

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3, 7, 9, and 12. Similarly, the generalized CCA models extend the standard CCA model, which has been studied rigorously in refs. 5, 6, and 8. Other rigorous accounts dealing with these models can be found in refs. 2, 4, 10, 13, and 15. Finally, results concerning the existence of nontrivial stationary distributions for random continuous-time versions of these models for certain parameter values are obtained in ref. 1.

First of all, the parameter k, which is the number of colors, determines what the state space X is (once the dimension d is set). For GH, the updating rule is as follows. Each $i \in \{1, 2,..., k-1\}$ automatically becomes $i+1 \pmod k$ at the next stage. However, a 0 at site x becomes a 1 at the next stage if and only if at least θ of the sites in its neighborhood $\mathcal{N}(x) \equiv x + \mathcal{N}$ are in state 1 (this is where the two other parameters $\mathcal{N}$ and θ enter). Otherwise, the 0 remains a 0.

For the CCA, an $i \in \{0, 1,..., k-1\}$ at site x becomes $i+1 \pmod k$ at the next stage if and only if at least θ of the sites in its neighborhood $\mathcal{N}(x)$ are in state $i+1 \pmod k$. Otherwise, site x remains in state i. Therefore each state for the CCA model behaves like the 0 state for the GH model. Throughout this paper, the term *uniform product measure* will mean the product measure on $\{0, 1,..., k-1\}^{\mathbf{Z}^d}$ with each marginal uniform on $\{0, 1,..., k-1\}$.

Definition 1.1. $\{x_1, x_2,..., x_n\}$ is called a *path* (relative to the parameter $\mathcal{N}$) if $x_{i+1} \in \mathcal{N}(x_i) \equiv x_i + \mathcal{N}$ for $i = 1,..., n-1$.

We now say a word about the *standard* GH model. This has three colors, threshold 1, and neighborhood set $\mathcal{N} = \{y: \|y\|_1 \leqslant 1\}$, the usual L^1 neighborhood (where of course $\|y\|_1$ is the sum of the absolute values of the coordinates of y). The standard CCA model is defined analogously. In ref. 3 it is proven that *uniform asymptotic local periodicity* holds in the following sense.

Theorem 1.2. For the standard GH model in $d \geqslant 2$ dimensions starting with uniform product measure, each lattice point is eventually periodic, cycling at period 3 a.s.

The proof of this is identical to the proof of the analogous theorem for CCA which was done earlier in ref. 8. The key idea is to note that any

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{matrix}$$

which sits in a two-dimensional sublattice of the configuration simply cycles at period 3 independent of the outside. Such cycles are examples of what are called *clocks*. If we initial distribution is uniform product measure, then the initial configuration will contain a clock somewhere a.s. From here, one proves Theorem 1.2 by showing that it is a deterministic fact that if a configuration contains such a clock somewhere, then every lattice point is eventually periodic, cycling at period 3.

Our first theorem here concerns this notion of uniform asymptotic local periodicity for a different set of parameter values where the argument is more involved.

Theorem 1.3. Consider the GH or CCA model in $d \geqslant 2$ dimensions with three colors, threshold 2, and neighborhood set $\mathcal{N} = \{y\colon \|y\|_\infty \leqslant 1\}$ (where $\|y\|_\infty$ is the maximum of the absolute values of the coordinates of y). Then, if we start the system with uniform product measure, each lattice point is eventually periodic, cycling at period 3 a.s.

We first note that under the parameters in Theorem 1.3,

$$\begin{array}{ccc} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{array}$$

cycles at period 3 independent of the outside. This therefore plays the role here (as could other finite configurations with this property) that

$$\begin{array}{ccc} 0 & 1 & 2 \\ 2 & 1 & 0 \end{array}$$

played in the standard GH and CCA models. However, the proof of Theorem 1.2 no longer works when the threshold is raised to 2 and $\mathcal{N}$ changes from $\{y\colon \|y\|_1 \leqslant 1\}$ to $\{y\colon \|y\|_\infty \leqslant 1\}$ and other methods are required. In particular, it is not a deterministic fact for this model that a configuration with the above structure in it is eventually periodic, cycling at period 3, at every lattice point. The following configuration for $d=2$ demonstrates this, as is easily verified, where ? can be taken to be any of 0, 1, or 2:

$$\begin{array}{ccccccccccccccc}
\ldots 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ldots \\
\ldots 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \ldots \\
\ldots 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 \ldots \\
\ldots ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \ldots \\
\ldots ? & ? & ? & ? & ? & ? & 1 & 0 & 1 & ? & ? & ? & ? & ? & ? \ldots \\
\ldots ? & ? & ? & ? & ? & ? & 2 & 2 & 2 & ? & ? & ? & ? & ? & ? \ldots \\
\ldots ? & ? & ? & ? & ? & ? & 1 & 0 & 1 & ? & ? & ? & ? & ? & ? \ldots \\
\ldots ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \ldots
\end{array}$$

It is believed that whenever the parameters are such that there exists a finite object which cycles at period k independent of the outside (see Definition 2.1), then every lattice point is eventually periodic (but not necessarily having period k), which one could call asymptotic (not necessarily uniform) local periodicity. There are cases where the above local periodicity is not uniform. Theorem 1.3 proves this conjecture for one set of parameter values together with a uniformity in the period. Computer simulations[11] indicated earlier that Theorem 1.3 is true.

Our second theorem is the following, stronger version of Theorem 1.2, which states that every lattice point is "eventually controlled by some finite clock" in a precise sense. We need the following definition.

Definition 1.4. A self-avoiding path $\{z_0, z_1,..., z_n = z_0\}$ (except for $z_n = z_0$) is a *clock* for η for standard GH or CCA if $\eta(z_{i+1}) = \eta(z_i) + 1$ (mod 3) for each i.

We note that n necessarily is a multiple o 6 in the above and that, as was the case with

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{matrix}$$

(which is itself of course a clock), it is trivial to see that a clock cycles at period 3 independent of the outside.

Theorem 1.5. For the standard GH or CCA model in two dimensions, a.s. (with respect to uniform product measure) for every $x \in \mathbf{Z}^2$, there will be a clock $\{y_1, y_2,..., y_{6n}\}$ for η_0, a time T, and a self-avoiding path $\{x = x_0, x_1,..., x_l\}$ that intersects the clock precisely at x_l such that $\eta_T(x_{i+1}) = \eta_T(x_i) + 1$ (mod 3) for $i = 0,..., l-1$.

We note that in the above, x will be periodic, cycling at period 3, after time T. In addition to of course implying Theorem 1.2, this gives a better description of what the final period-3 configuration looks like.

With regard to using percolation in cellular automata, we mention ref. 12, in which percolation is used in an important way to analyze a cellular automation.

The remainder of the paper is devoted to proofs.

2. PROOFS

Throughout this section, we will only discuss the GH model, since the proofs of the results for CCA are identical. We first generalize the notion of a clock that we introduced in Section 1 for the case of the standard GH model, a notion which we steal from ref. 10.

Definition 2.1. A *stable periodic object (spo)* (relative to the parameters k, $\mathcal{N}$, and θ) for a configuration η is a finite set S of lattice points $\{x_1,..., x_n\}$ such that for all i,

$$|\{y \in \mathcal{N}(x_i) \cap S: \eta(y) = \eta(x_i) + 1 \pmod k\}| \geqslant \theta$$

Note that S being an spo for η depends only on the restriction of η to S. The following lemma is obvious and explains the terminology "stable periodic object."

Lemma 2.2. If S is an spo for a configuration η, then (under the Greenberg–Hastings dynamics with the relevant parameters) independent of the values of η outside S, lattice points in S will cycle at period k, increasing its value one unit (mod k) each time.

Proof of Theorem 1.3. We carry out the proof only for $d=2$. The extension to $d \geqslant 3$ is trivial and left to the reader. As indicated in the introduction, the lattice points where the finite configuration

$$\begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{matrix} \tag{2.1}$$

sits is an spo. Of course, since our initial distribution is uniform product measure, such an spo will exist somewhere in the initial configuration a.s. In view of the example given in Section 1, we cannot conclude that uniform asymptotic local periodicity is then achieved as a deterministic fact and so more probability is needed for the argument.

Let Y_0 be the union of all lattice points which are contained in an spo of the form (2.1) in the initial configuration η_0. By Lemma 2.2, all lattice points in Y_0 from time 0 cycle at period 3 independent of the states of the rest of the lattice points. Let $Y_1 = Y_0 \cup \{x: |\mathcal{N}(x) \cap Y_0| \geqslant 4\}$. Inductively, we let $Y_{k+1} = Y_k \cup \{x: |\mathcal{N}(x) \cap Y_k| \geqslant 4\}$ and then let $Y_\infty = \bigcup_{i=0}^{\infty} Y_i$.

We claim that any x in Y_∞ eventually cycles at period 3, which is easily proved by induction on k. Let x be in Y_{k+1} and assume that all lattice points in Y_k eventually cycle at period 3. If $x \notin Y_k$, then four neighbors of x will be in Y_k and so, by assumption, by some time T, these four neighbors will be periodic, cycling at period 3. At this time T, two of these four neighbors, say z_1 and z_2, are always the same color, since there are only three colors. Now consider $\eta(z_1) - \eta(x)$ (mod 3) from time T onward. If x increases its value (mod 3), then this quantity remains unchanged, while if x stays fixed, then this quantity increases by 1 (mod 3). If x does not eventually cycle at period three, it will stay at least two time units at 0 infinitely often. By considering the quantity $\eta(z_1) - \eta(z)$ (mod 3) above, it follows that there will then be some time after time T when $\eta(z_i) = \eta(x)+1 \pmod 3$. At this point, x will also begin to cycle at period 3, since z_1 and z_2 will allow x to get from state 0 to state 1, giving a contradiction.

We therefore need to show that $Y_\infty = Z^2$ a.s. let $B = Y_\infty^c$, which is contained in Y_0^c. Note that by its construction, B has the strange property that for all x in B, $|\mathcal{N}'(x) \cap B| \geq 5$, where $\mathcal{N}'(x) = \mathcal{N}(x) \backslash \{x\}$. (There are eight neighbors altogether, so if x had fewer than five neighbors in B, then it would have at least four in Y_∞ and hence would have also been in Y_∞.)

We call a set S *5-thick* if for all x in S, $|\mathcal{N}'(x) \cap S| \geq 5$. So B above is a 5-thick set contained in Y_0^c. To finish the proof, we need to prove the following percolation proposition and apply it to the set B. ∎

Propostion 2.3. The probability that there is a nonempty 5-thick set contained in Y_0^c is 0.

The next definition formulates the idea of a path around the origin which does not "stick in" anywhere or is (together with its interior) convex. What makes the proof of Proposition 2.3 work is that the number of such paths around 0 of length l is polynomial (as opposed to exponential) in l.

Definition 2.4. A *convex path around 0* is a path (see Definition 1.1) $x_0, x_1,..., x_n = x_0$ which is self-avoiding (except for $x_n = x_0$) that goes clockwise around 0 (i.e., has winding number 1 around 0) and such that the induced path in R^2 (which is $x_0, x_1,..., x_n = x_0$ together with the line segments in the plane connecting subsequent points) together with the area in the plane that the path surrounds is a convex set in R^2.

Lemma 2.5. Let S be a nonempty 5-thick set such that S contains no infinite rays and $0 \notin S$. Then S contains a convex path around 0.

Proof. If $S \cap \{(x, y)\colon y = 0\} = \varnothing$, let $p = (p_x, p_y)$ be any point in $S \cap \{(x, y)\colon y > 0\}$ with minimum y coordinate. Since S is 5-thick and all points on the horizontal line immediately below p are not in S, it is easy to see that all points with the same y coordinate as p must be in S, contradiction the fact that S contains no infinite rays.

If $S \cap \{(x, y)\colon y = 0\} \neq \varnothing$, we may assume that there is a point lying to the left of 0 (on the x axis) and we let z_0 be the closest point to 0 on the left. Let z_1 be the first element of $(z_0 + (1, 1), z_0 + (0, 1), z_0 + (-1, 1))$ which is in S [as S is 5-thick and $z_0 + (1, 0) \notin S$, one of these three points must be in S]. We construct a sequence $(z_0, z_1, z_2,...)$ inductvely as follows. Intuitively, we build our path by going clockwise around 0, staying in S, always trying to move inward as much as possible.

Consider the following ordered set of vectors:

$$(1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0), (-1, 1), (0, 1), (1, 1)$$

which we call $(v_0, v_1, ..., v_7)$, where we consider them cyclically in the sense that $(1, 0)$ follows $(1, 1)$. These correspond to the directions the path we will construct can go in. To define z_{i+1}, we consider the vector $z_i - z_{i-1}$, which we assume is v_j in the above list. We then take z_{i+1} to be the first element of $(z_i + v_{j+2}, z_i + v_{j+1}, z_i + v_j)$ which is in S [where the indices $j, j+1$, and $j+2$ are of course taken (mod 8)]. As S is 5-thick and by the way we are building our path, one of these three will be in S.

Since S contains no infinite rays, we will never get trapped into infinitely often choosing the last element of the above triple (which would correspond to continuing in the same direction forever). Therefore by observing that we could also start from z_0 and go downward [by choosing the first of $z_0 + (1, -1)$, $z_0 + (0, -1)$, and $z_0 + (-1, -1)$ which is in S] instead of upward, there must eventually be some repeated site (i.e., we cannot spiral out to ∞ without repeating a lattice point).

Since we have always taken the first element of the above triple, it is clear that z_0 is the first repeated site and that the path we have constructed (by stopping when we return to z_0) is the desired convex path around 0. ∎

Lemma 2.6. $\text{Prob}((0 \in Y_0) \cap F) > 0$, where F is the event that there is no convex path around 0 contained in Y_0^c.

Proof. For N with $2N+1$ a multiple of 3, let E_N be the event that the obvious tiling of $[-N, N]^2$ (where we mean of course by $[-N, N]^2$ the usual set $[-N, N]^2$ in the plane intersected with the 2D integer lattice) by 3×3 squares has the property that

$$\begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{matrix}$$

appears on each of these 3×3 squares in the initial configuration η_0. Since $E_N \cap F \subseteq (0 \in Y_0) \cap F$ and $\text{Prob}(E_N) > 0$ for all N, we need to show that for some N, $\text{Prob}(F^c \mid E_N) < 1$.

If γ is a convex path around 0 of length L not intersecting $[-N, N]^2$, then we can find $L/10$ points on γ such that each of these points has a 3×3 square not intersecting $[-N, N]^2$ in which it sits and such that these 3×3 squares are pairwise disjoint. Now, for this fixed γ, if $\gamma \subseteq Y_0^c$, then on each of these $L/10$ 3×3 squares, the configuration is necessarily not

$$\begin{matrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{matrix}$$

Since the 3×3 squares are pairwise disjoint and disjoint from $[-N, N]^2$, it follows that

$$\mathrm{Prob}(\gamma \subseteq Y_0^c \mid E_N) \leqslant [1 - (\tfrac{1}{3})^9]^{L/10}$$

Next, the number of convex paths of length L around 0 is at most L^{10} (an upper bound easily verified by looking at the places where the convex path changes its direction). It follows that

$$\mathrm{Prob}(F^c \mid E_N) \leqslant \sum_{L \geqslant N} L^{10}[1 - (\tfrac{1}{3})^9]^{L/10}$$

We now simply choose N large enough so that this sum is <1. ▮

Proof of Proposition 2.3. Calling this event E, it suffices by ergodicity to show that $\mathrm{Prob}(E) < 1$. Letting G be the event that Y_0^c contains no infinite ray, we clearly have $\mathrm{Prob}(G) = 1$, which implies by Lemma 2.6 that $\mathrm{Prob}((0 \in Y_0) \cap F \cap G) > 0$, where F is defined in Lemma 2.6. Next Lemma 2.5 tells us that $(0 \in Y_0) \cap F \cap G \subseteq E^c$, which implies $\mathrm{Prob}(E) < 1$, as desired. ▮

Before giving the proof of Theorem 1.5, it is useful to make the following definition.

Definition 2.7. We say that x_0 *is connected to* ∞ *for* η if there is a self-avoiding path $x_0, x_1, x_2,...$ such that $\eta(x_{i+1}) = \eta(x_i) + 1 \pmod 3$ for each i.

Proof of Theorem 1.5. By Theorem 1.2, $\eta_{3n} \equiv T^{3n}\eta_0$ converges a.s. (in the product topology), where η_0 is chosen from uniform product measure. Let η_∞ denote this random limit. We mention again that if x is cycling at period 3 and y is a neighbor of x with $\eta(x) = \eta(y) + 1 \pmod 3$, then y is also cycling at period 3.

Let B_x be the event described in the theorem whose probability is claimd to be 1, where we write B for B_0. By translation invariance, we need to show that $\mathrm{Prob}(B) = 1$. We now define A to be the event that 0 is connected to ∞ for the configuration η_∞. We will show $\mathrm{Prob}(B) = 1$ by proving that $\mathrm{Prob}(A \cup B) = 1$ and $\mathrm{Prob}(A) = 0$.

We first note that for every $x \in \mathbf{Z}^2$, there is a neighbor y of x such that $\eta_\infty(y) = \eta_\infty(x) + 1 \pmod 3$. This is clear since otherwise when x becomes 0, it would not have a 1 next to it to ensure its advancing by 1. Next, construct a sequence of lattice points as follows. Let $x_0 = 0$, x_1 be a neighbor of x_0 such that $\eta_\infty(x_1) = \eta_\infty(x_0) + 1 \pmod 3$, x_2 be a neighbor of x_1 such that $\eta_\infty(x_2) = \eta_\infty(x_1) + 1 \pmod 3$, and so on. If all these points are distinct, we are then in event A. If, on the other hand, there is a repeat, let x_i and x_j be the first pair of points in the sequence which are the same.

Then $(x_i, x_{i+1},..., x_{j-1})$ is clearly a clock for η_∞ (see Definition 1.4). One can show (see ref. 3 or the earlier ref. 8 for the analogous result for CCA) that this implies that $(x_i, x_{i+1},..., x_{j-1})$ is also a clock for η_0. The argument is as follows. If $(x_i, x_{i+1},..., x_{j-1})$ is a clock for η_∞, then clearly $(x_i, x_{i+1},..., x_{j-1})$ is also a clock for η_l for some large l. We next claim that it follows that $(x_i, x_{i+1},..., x_{j-1})$ is also a clock for η_{l-1}, which gives the desired result by induction. Certainly all the 2's (1's) in $(x_i, x_{i+1},..., x_{j-1})$ at time l are 1's (0's) at time $l-1$. Now, if some 0 at time l in $(x_i, x_{i+1},..., x_{j-1})$ was a 0 at time $l-1$, then it would have sat next to a 1 at time $l-1$ (since it sat next to a 2 at time l) and hence would have been a 1 at time l, giving us a contradiction. Hence the 0 at time l must have been a 2 at time $l-1$, showing that $(x_i, x_{i+1},..., x_{j-1})$ was a clock at time $l-1$. This shows that $(x_i, x_{i+1},..., x_{j-1})$ is a clock for η_0, as desired. We are therefore in event B, where we take T to be large enough so that $x_0, x_1,..., x_{j-1}$ have all reached periodicity by time T. Hence $\text{Prob}(A \cup B) = 1$.

To show $\text{Prob}(A) = 0$, we now need the following lemma, whose proof will be given later.

Lemma 2.8. Let U be the event that 0 is connected to ∞ for the configuration η_0. Then $\text{Prob}(A) = 0$ if and only if $\text{Prob}(U) = 0$.

The event U is a type of dependent oriented bond percolation problem that can be formulated as follows. Throw down 0's, 1's, and 2's at random on the lattice with a uniform product measure. An arrow is then draw from a to b if $\|a-b\|_1 = 1$ and $b = a+1 \pmod 3$. Clearly, for each pair a and b with $\|a-b\|_1 = 1$, there is an arrow from a to b with probability 1/3, there is an arrow from b to a with probability 1/3, and there is no arrow at all between a and b with probability 1/3. Then U is the event that there is a self-avoiding oriented path from 0 to ∞. It is well known (and easy to prove) that the number of self-avoiding paths of length n in two dimensions starting from the origin, which we call a_n, is $\leqslant Kc^n$ for positive constants K and c with $c < 3$. Next, it is clear that although the arrows are not independent, the probability that a particular self-avoiding path γ of length n starting from the origin has the property that η_0 increases one value (mod 3) at each step as we transverse γ starting from the origin is $1/3^n$. Hence $\text{Prob}(U) \leqslant a_n \cdot 1/3^n \leqslant Kc^n \cdot 1/3^n$ for all n. As $c < 3$, we have that $\text{Prob}(U) = 0$ and so $\text{Prob}(A) = 0$ by Lemma 2.8, as desired. ▌

Proof of Lemma 2.8. Clearly, $U \subseteq A$ and so one direction is easy. We assume that $\text{Prob}(U) = 0$. If 0 is connected to ∞ for η_n, one can easily show that there must be some lattice point x (with L^1 distance at most n from 0) such that x is connected to ∞ for η_0. Now this latter event occurs with probability 0 by the translation invariance and the fact that Prob(U) = 0. Hence Prob(0 is connected to ∞ for η_n) = 0 for all n. We now only need to show that $A \subseteq \bigcup_{n=0}^{\infty}$ {0 is connected to ∞ for η_n}.

Assume there exists a self-avoiding path $0 = x_0, x_1, \ldots$ such that $\eta_\infty(x_{i+1}) = \eta_\infty(x_i) + 1 \pmod 3$ for all i. By Theorem 1.2, there is a.s. an integer N such that lattice point 0 cycles at period 3 from time N onward. We claim then that 0 is connected to ∞ for η_N.

To see this, we know that 0 is cycling at period 3 after time N. If $\eta_N(x_1)$ were one less (mod 3) than $\eta_N(x_0)$, it would stay one less, contradicting $\eta_\infty(x_1) = \eta_\infty(x_0) + 1$. If $\eta_N(x_1)$ were there same as $\eta_N(x_0)$, then the two will either stay the same as each other or the value at x_1 will drop one behind and stay one behind, in either case again contradicting $\eta_\infty(x_1) = \eta_\infty(x_0) + 1$. So $\eta_N(x_1)$ must be one higher than $\eta_N(x_0)$ and the same argument shows it must also always stay one ahead and hence must also be cycling at period 3. By induction, one gets that $\eta_N(x_{i+1}) = \eta_N(x_i) + 1 \pmod 3$ for all i and so 0 is connected to ∞ for η_N, as desired. ∎

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